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The Koszul formulae for graded Lie - Cartan pairs (Super BRS operator)

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Abstract. *This note is a sequel to the paper* [1] *codifying the* $\mathbb{Z}/2$*-graded (i.e. fermionic) differential calculus within the frame of «graded Lie-Cartan pairs». We here present in this frame the Koszul formulae for the generalized exterior derivative, resp. the generalized covariant exterior derivative. The latter yields in particular the cohomological aspect of the «super BRS operator».*

§ 1. INTRODUCTION

In the sequel of § 1 we recall the facts about graded Lie Cartan pairs (cf. [1] for details) which are needed for displaying the Koszul formulae treated in § 2.

We recall that a *graded Lie-Cartan pair* is a pair (L, A) of a Lie superalgebra L, and a supercommutative (1) algebra A, with, for $\xi \in L$ and $a \in A$, two bilinear products $\xi a \in A$ and $a\xi \in L$, the first corresponding to an action of L as derivations (2) of A:

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(1) I.e. $A = A^0 \oplus A^1$ with $A^iA^j \subset A^{i+j}$, and $ba = (-1)^{ij}ab$, $a \in A^i$, $b \in A^j$, $i, j \in \mathbb{Z}/2$. A is supposed to have a unit 1_A.

(2) In the graded sense, i.e. sums of even derivations and odd antiderivations. (1) defines a homomorphism: $A \to Der\,A$ of Lie superalgebras.

(1.1) $\quad A \quad a \to \xi a \in A,\ \xi \in L;$

whilst the second yields a left unital A-module structure (3) of L:

(1.2) $\quad \xi \in L \to a\xi \in L,\ a \in A;$

this moreover with the two properties

(1.3) $\quad a(\xi b) = (a\xi)b,\ a, b \in A,\ \xi \in L,$

and (4)

(1.4) $\quad [\xi, a\eta] = (-1)^{\partial\xi\partial a} a[\xi, \eta] + (\xi a)\eta,\ \xi \in L^{\cdot}, \eta \in L,\ a \in A^{\cdot}.$

The generalized exterior derivative δ is as follows: it acts on the direct sum of graded-alternate n-forms on L with values in A;

(1.5)
$$\Lambda^*(L, A) = \bigoplus_{n \in \mathbb{N}} \Lambda^n(L, A),$$
$$(\Lambda^0(L, A) = A),$$

equipped with the graded wedge product (5) $\wedge$, and with the total grading of forms, where the *graded wedge product is given as follows in low N-grade:*

(1.6)
$$\begin{cases} (a \wedge \varphi)(\xi) = (-1)^{\partial a} a\varphi(\xi), \begin{cases} a \in \Lambda^0(L, A)^{\bullet} = A^{\bullet}, \varphi \in \Lambda^1(L, A) \\ \xi \in L \end{cases}, \\ (-1)^{\partial_0\varphi_1}(\varphi_1 \wedge \varphi_2)(\xi_1, \xi_2) = (-1)^{\partial_0\varphi_2\partial\xi_1}\varphi_1(\xi_1)\varphi_2(\xi_2), \begin{cases} \varphi_1, \varphi_2 \in \Lambda^1(L, A)^{\bullet} \\ \xi_1, \xi_2 \in L^{\bullet} \end{cases}, \\ \qquad - (-1)^{\partial\xi_2(\partial\xi_1 + \partial_0\varphi_2)}\varphi_1(\xi_2)\varphi_2(\xi_1) \end{cases}$$

whilst the *total grade* of $\alpha \in \Lambda^n(L, A)$ is $\partial\alpha = n + \partial_0\alpha$, $\partial_0\alpha$ the *intrinsic grade,* given by

(1.7)
$$\begin{cases} \partial_0\alpha = \partial\, \alpha(\xi_1, \dots, \xi_n) - \sum_{i=1}^{n} \partial\xi_i,\ \xi_1, \dots, \xi_n \in L^{\bullet}. \\ \text{whenever } \alpha(\xi_1, \dots, \xi_n) \neq 0. \end{cases}$$

(3) I.e. $a\,\xi$ is bilinear, $b(a\xi) = (ba)\xi$, and $1_A\xi = \xi$, $\xi \in L$, $a, b \in A$.

(4) For a graded vector space $E = E^0 \oplus E^1$, we denote by $E^{\cdot}$ the set $E^0 \cup E^1$ of homogeneous elements and by ∂x the grade of $x \in E^{\cdot}$ ($x = i \bmod 2$ for $x \in E^i$).

(5) Making $\Lambda^*(L, A)$ a supercommutative algebra for the total grading (cf. [1]). For the general expression of the graded wedge product, see [1], (4.3, 4).

δ is then defined as follows: it is the sum $\delta = \delta_0 + d\wedge$ of two grade-one derivations (6) of $\Lambda^*(L, A)$ of N-grade one specified by their actions in N-grade 0 and 1 given as follows: one has, for $a \in \Lambda^0(L, A)^\cdot = A^\cdot$, $\alpha \in \Lambda^\cdot(L, A)$, and $\xi_1, \xi_2 \in L^\cdot$:

$$(1.8)\qquad \begin{cases} \delta_0 a = 0 \\ \delta_0 \alpha(\xi_1, \xi_2) = -\alpha([\xi_1, \xi_2]) \end{cases}$$

$$(1.9)\qquad \begin{cases} (d\wedge a)(\xi_1) = (-1)^{\partial\xi_1 \partial_0 a}\xi_1 a \\ (d\wedge \alpha)(\xi_1, \xi_2) = (-1)^{\partial\xi_1 \partial_0 \alpha}\xi_1\{\alpha(\xi_2)\} \\ \qquad\qquad - (-1)^{\partial\xi_2(\partial_0\alpha + \partial\xi_1)}\xi_2\{\alpha(\xi_1)\} \end{cases}$$

The requirements (7) and (8) determine δ_0 and $d\wedge$ stepwise from $\Lambda^n(L, A)$ to $\Lambda^{n+1}(L, A)$; δ is a *differential* in the sense that $\delta^2 = 0$ (7). In addition to δ we shall need the *inner product* $i(\xi)$ by $\xi \in L$ acting on $\Lambda^*(L, A)$ as follows: it vanishes on $\Lambda^0(L, A) = A$, and we have, for $\xi \in L^0$, $\alpha \in \Lambda^n(L, A)$, $n \geq 1$, $\xi_1, \dots, \xi_n \in L$.

$$(1.10)\qquad \{i(\xi)\alpha\}(\xi_1, \dots, \xi_{n-1}) = (-1)^{\partial\xi\partial\alpha}(\xi, \xi_1, \dots, \xi_{n-1})$$

$i(\xi)$ is a derivation of $\Lambda^*(L, A)$ of (total) grade $1 + \partial\xi$ in the sense that

$$(1.11)\qquad i(\xi)(\alpha\wedge\beta) = \{i(\xi)\alpha\}\wedge\beta + (-1)^{(1+\partial\xi)\partial\alpha}\alpha\wedge i(\xi)\beta, \begin{cases} \alpha \in \Lambda^p(L, A)^\bullet \\ \beta \in \Lambda^q(L, A) \end{cases}$$

The *generalized covariant derivative* δ_ρ is, on the other hand, as follows: one specifies a graded unital right A-module (8) E equipped with an *E-connection* (i.e. a grade zero linear map ρ from L into the graded ξ-derivations of E such that

$$(1.12)\qquad \rho(\xi)(Xa) = \{\rho(\xi)X\}a + (-1)^{\partial\xi\partial X}X(\xi a),\ X \in E^\bullet,\ a \in A$$

$$(\rho(L^i)X^j \subset X^{i+j},\ i, j \in \mathbf{Z}/2),$$

with the corresponding *curvature*

$$(1.13)\qquad \Omega_\rho(\xi, \eta) = [\rho(\xi), \rho(\eta)] - \rho([\xi, \eta])$$

(6) An odd derivation of $\Lambda^*(L, A)$ of N-grade one is a linear operator D of total grade one mapping $\Lambda^n(L, A)$ into $\Lambda^{n+1}(L, A)$, $n \in \mathbf{N}$, fulfilling

$$D(\alpha_1 \wedge \alpha_2) = (D\alpha_1)\wedge\alpha_2 + (-1)^{p\partial\alpha_1}\alpha_1\wedge D\alpha_2, \alpha_1, \alpha_2 \in \Lambda^*(L, A)^\cdot).$$

Such derivations are determined by their actions in grade 0 and 1.

(7) Hence $(\Lambda^*(L, A), \wedge, \delta)$ becomes a graded commutative differential algebra.

(8) i.e. $E = E^0 \oplus E^1$ is a graded vector space on which A acts linearly on the right with $E^iA^j \subset E^{i+j}$, $i, j \in \mathbf{Z}/2$ and $(\xi a)b = \xi(ab)$, $1_A\xi = \xi$, $\xi \in E$, $a, b \in A$.

(expressing the failure of ρ to be a representation of the Lie algebra L). The connection ρ is called *flat* whenever $\Omega = 0$, and *local* whenever $\rho(a\cdot\xi) = a\rho(\xi)$, $\xi \in L$, $a \in A$.

One then builds the space on which δ_ρ acts as the direct sum of spaces of graded alternate (9) n-forms on L with values in E:

(1.14) $$\Lambda^*(L, E) = \bigoplus_{n \in \mathbb{N}} \Lambda^n(L, E)$$

$$(\Lambda^0(L, E) = E)$$

also definable as the tensor product (10)

(1.14.a) $$\Lambda^*(L, E) = E \otimes_A \Lambda^*(L, A)$$

(via the identification

(1.15) $$(X \otimes \alpha)(\xi_1, \dots, \xi_n) = (-1)^{n\partial X} X\alpha(\xi_1, \dots, \xi_n)$$

$$X \in E^\bullet,\ \alpha \in \Lambda^n(L, A),\ \xi_1, \dots, \xi_n \in L)$$

and thus becoming a right $\Lambda^*(L, A)$-module, with

(1.16) $$(X \otimes \alpha)\beta = X \otimes (\alpha \wedge \beta),\ X \in E,\ \alpha, \beta \in \Lambda^*(L, A).$$

δ_ρ is then defined as the sum $\delta_\rho = \delta_0 + \rho \wedge$ of the odd δ-derivations of N-grade one (11) δ_0, and $\rho \wedge$ of $\Lambda^*(L, E)$, respectively specified by the following actions in grade 0 and 1 : one has, for $\lambda_0 \in \Lambda^0(L, E)^\bullet = E^\bullet$, $\lambda_1 \in \Lambda^1(L, E)^\bullet$, and $\xi_1, \xi_2 \in L^\bullet$

(1.17) $$\begin{cases} \delta_0\lambda_0(\xi_1) = 0 \\ \delta_0\lambda_1(\xi_1, \xi_2) = -\lambda_1([\xi_1, \xi_2]) \end{cases}$$

resp.

(9) See [1], Appendix B, for the definition of graded alternate forms.

(10) Quotient of the usual tensor product by the relation $(Xa) \otimes \alpha = X \otimes (a \wedge \alpha)$, $X \in E$, $\alpha \in \Lambda^*(L, A)$, $a \in A$.

(11) An odd η-derivation of $\Lambda^*(L, E)$ of N-grade one, η an odd derivation of $\Lambda^*(L, A)$ of N-grade one, is a linear operator D of total grade one mapping $\Lambda^n(L, E)$ into $\Lambda^{n+1}(L, E)$, fulfilling

$$D(\lambda\alpha) = (D\lambda)\alpha + (-1)^{\partial\lambda}\lambda(\eta\alpha),\ \lambda \in \Lambda^p(L, E)^\bullet,\ \alpha \in \Lambda^q(L, A).$$

Such a D is determined by its action in grade 0 and 1.

For explicit expressions of the actions of $\rho\wedge$ and δ_0 on n-forms, we refer to [1], (3.1, 2, 3).

$$(1.18)\qquad \begin{cases} (\rho \wedge \lambda_0)(\xi_1) = (-1)^{\partial \xi_1 \partial \lambda_0} \rho(\xi_1)\lambda_0 \\ (\rho \wedge \lambda_1)(\xi_1, \xi_2) = (-1)^{\partial \xi_1 \partial_0 \lambda_1} \rho(\xi_1)\{\lambda_1(\xi_2)\} \\ \qquad\qquad - (-1)^{\partial \xi_2 (\partial_0 \lambda_1 + \partial \xi_1)} \rho(\xi_2)\{\lambda_1(\xi_1)\} \end{cases}$$

We shall also need the *inner product* $i(\xi)$ by $\xi \in L$ acting on $\Lambda^*(L, E)$ as follows: it vanishes on $\Lambda^0(L, E) = E$, and we have, for $\xi \in L^0$, $\lambda \in \Lambda^n(L, E)$, $n \leq 1$, $\xi_1, \ldots, \xi_n \in L$:

$$(1.19)\qquad \{i(\xi)\lambda\}(\xi_1, \ldots, \xi_{n-1}) = (-1)^{\partial \xi \partial \lambda} \lambda(\xi, \xi_1, \ldots, \xi_{n-1})$$

Now $i(\xi)$ is a ξ-derivation of (total) grade $1 + \partial\xi$ of the module $\Lambda^*(L, E)$:

$$(1.20)\qquad i(\xi)(\lambda\alpha) = \{i(\xi)\lambda\}\alpha + (-1)^{(1+\partial\xi)\partial\lambda} \lambda\{i(\xi)\alpha\}, \begin{cases} \lambda \in \Lambda^*(L, E)^{\cdot} \\ \alpha \in \Lambda^*(L, A) \end{cases}$$

Note that δ and $i(\xi)$ in (1.10) are in fact the special cases of δ_ρ and $i(\xi)$ in (1.19) corresponding to the choice of A itself as the left A-module E, with $\rho(\xi)a = \xi a$, $\xi \in L$, $a \in A$.

Finally we shall need the operators $d(\xi)$, resp. $\rho(\xi)$, $\xi \in L^\bullet$, acting on $\Lambda^*(L, A)$, resp. $\Lambda^*(L, E)$ as follows: for $\xi \in L^\bullet$, $\alpha \in \Lambda^*(L, A)$, $\lambda \in \Lambda^*(L, E)$, and $\xi_1, \ldots, \xi_n \in L$, we have

$$(1.21)\qquad \{d(\xi)\alpha\}(\xi_1, \ldots, \xi_n) = (-1)^{n\partial\xi} \rho(\xi)\{d(\xi_1, \ldots, \xi_n)\}$$

and

$$(1.22)\qquad \{\rho(\xi)\lambda\}(\xi_1, \ldots, \xi_n) = (-1)^{n\partial\xi} \rho(\xi)\{\lambda(\xi_1, \ldots, \xi_n)\}$$

(note that $d(\xi)$ is a special case of $\rho(\xi)$ corresponding to $E = \mathbb{C}$ and ρ the map (1, 2), to which $d(\xi)$ reduces in order zero). $\rho(\xi)$ is a $d(\xi)$-derivation of (total) grade $\partial\xi$ of the module $\Lambda^*(L, E)$: one has

$$(1.23)\qquad \rho(\xi)(\lambda\alpha) = \{\rho(\xi)\lambda\}\alpha + (-1)^{\partial\xi\partial\lambda} \lambda\{d(\xi)\alpha\}, \begin{cases} \lambda \in \Lambda^*(L, E)^\bullet \\ \alpha \in \Lambda^*(L, A) \end{cases}$$

which in the particular case $E = \mathbb{C}$, $\rho = d$, reduces to

$$(1.24)\qquad d(\xi)(\alpha \wedge \beta) = \{d(\xi)\alpha\} \wedge \beta + (-1)^{\partial\xi\partial\alpha} \alpha \wedge \{d(\xi)\beta\}, \begin{cases} \alpha \in \Lambda^*(L, A)^\bullet \\ \beta \in \Lambda(L, A) \end{cases}$$

§2. THE KOSZUL FORMULAE

We now consider the special case where L is a *finetely generated free*

A-module (12), i.e. possesses a *dual base* $(e_i, \epsilon^i)_{i=1,\dots,r}$, consisting of k elements $e_i \in L$, and k A-linear forms $\epsilon^i : E \to A$, fulfilling the duality and completeness relations (13):

$$\epsilon^i(e_k) = \delta^i_k \, 1_A, \quad i, k = 1, \dots, r \tag{2.1}$$

$$\sum_{i=1}^{r} e_i \epsilon^i = id_E \tag{2.2}$$

$$\left(\text{i.e. } \xi = \sum_{i=1}^{r} e_i \epsilon^i(\xi) = \sum_{i=1}^{r} (-1)^{\partial e_i(1+\partial \xi)} \epsilon^i(\xi) e_i, \quad \xi \in L^\bullet\right)$$

Then there is a useful expansion of the above operators δ_0, $d\wedge$, resp. δ_0, $\rho\wedge$ in terms of the dual basis, given by:

PROPOSITION (The Koszul Formulae). *Let (L, A) be a graded Lie Cartan pair such that the A-module L is free, with dual base $(e_i, \epsilon^i)_{i=1,\dots,r}$ consisting of homogeneous elements (so that* (14)

$$\partial_0 \epsilon^i = \partial e_i, \quad i = 1, \dots, r), \tag{2.3}$$

and let ρ be a local E connection. Then, with $c^k_{j,i}$ the structure constants of L in the basis e_i:

$$\begin{aligned} &[e_i, e_j] = \sum_{k=0}^{r} e_k c^k_{ij} = \sum_{k=0}^{r} (-1)^{\partial e_k(1+\partial e_i + \partial e_j)} c^k_{ij} e_k \\ &(\text{i.e. } c^k_{ij} = \epsilon^k([e_i, e_j]), \; i, j, k = 1, \dots, r, \end{aligned} \tag{2.4}$$

we have the following formulae (15) *for the generalized covariant derivative $\delta = \delta_0 + d\wedge$ resp. generalized covariant exterior derivative $\delta_\rho = \delta_0 + \rho\wedge$*

(12) Always the case if $A = \mathbb{C}$.

(13) The completeness relation is understood in the sense that $\sum_{i=1}^{r} e_i \epsilon^i(\xi) = \xi$ for all $\xi \in L$ ($e_i\epsilon^i$, the composition of the maps $\epsilon^i: E \to A$ and $e_i: A \to E$ as the map: $A \ni a \to e_i a \in E$), with E turned into a right A-module by the convention $Xa = (-1)^{\partial X \partial a} aX$, $X \in E^\bullet$, $a \in A^\bullet$.

(14) As follows from (2.1) and definition (1.7).

(15) Note that (2.5) is the special case of (2.6) corresponding to $E = \mathbb{C}$, $\rho = d$. In fact the locality of the E-connection ρ is needed only for proving the second line in (2.6). The first line (2.6) and (2.7) hold for a general ρ.

$$\delta_0 = \sum_{i,j,k=1}^{r} \frac{(-1)^{\partial e_i}}{2} c_{ji}^k (\epsilon^i \wedge)(\epsilon^j \wedge)\, i(e_k) \tag{2.5}$$

$$d\wedge = \sum_{i=1}^{r} (-1)^{\partial e_i} (\epsilon^i \wedge)\, d(e_i)$$

resp.

$$\delta_0 = \sum_{i,j,k=1}^{r} \frac{(-1)^{\partial e_i}}{2} c_{ji}^k (\epsilon^i \bullet)(\epsilon^j \bullet)\, i(e_k) \tag{2.6}$$

$$\rho \wedge = \sum_{i=1}^{r} (-1)^{\partial e_i} (\epsilon^i \bullet)\, \rho(e_i)$$

(here $(\alpha\wedge)$, $\alpha \in \Lambda^*(L, A)$ *denotes the operator: wedging from the left in* $\Lambda^*(L, A)$ *by the element* α*; whilst* $(\alpha\bullet)$ *denotes the left multiplication by* α *in* $\Lambda^*(L, E)$ *considered as a left* $\Lambda^*(L, A)$*-module-cf. footnote* (13)).

Note that, in the case of the «degenerate» Lie-Cartan pair $(L, \mathbb{C})$, L a Lie (super)algebra (where we have $\xi a = 0$, $\xi \in L$, $a \in \mathbb{C}$), δ is the *coboundary operator of the cohomology of L;* whilst δ_ρ is the *coboundary operator of the Chevalley cohomology of L with values in the representation space E* (observe that, in that case, we have $\Omega_\rho = 0$, hence ρ is a Lie-(super)algebra representation – cf. (1.3) above). Formulae (2.6) then yields the ($\mathbb{Z}/2$-graded generalization) of the familiar form of the BRS operator (cf. e.g. [3] Vol. 1, (3, 2, 4)), where the $id \otimes (\epsilon^i \wedge)$ correspond to the ghosts, and the $i(e_k)$ to the antighosts).

For the proof of the Proposition, we will need the

LEMMA. *With the assumptions and notation in the above Proposition, we have that, for i, j, k,* $\alpha, \beta = 1, \ldots, r$

$$\delta\epsilon^k = \sum_{i,j=1}^{r} \frac{(-1)^{\partial e_i}}{2} c_{ji}^k \epsilon^i \wedge \epsilon^j = \tag{2.7}$$

$$= \left(\sum_{i,j=1}^{r} \frac{(-1)^{\partial e_j}}{2} c_{ji}^k \epsilon^i \wedge \epsilon^j \right)$$

stemming from the facts that

$$\delta\epsilon^k (e_i, e_j) = -c_{ij}^k 1_A, \tag{2.8}$$

and

$$(2.9) \qquad (-1)^{\partial e_\alpha}(\epsilon^\alpha \wedge \epsilon^\beta)(e_i, e_j) = \{(-1)^{\partial e_i \partial e_j}\delta_i^\alpha \delta_j^\beta - \delta_j^\alpha \delta_i^\beta\}1_A,$$

hence

$$(2.10) \qquad \sum_{\alpha,\beta=1}^{r} (-1)^{\partial e_\alpha} c_{\beta\alpha}^k (\epsilon^\alpha \wedge \epsilon^\beta)(e_i, e_j) = -2c_{ij}^k 1_A$$

Proof of the Lemma. (2.7) follows from (2.8) and (2.10). Now (2.8) is obtained by making $\alpha = \epsilon^k$, $\xi_1 = e_1$, $\xi_2 = e_2$ in the second equation (1.8) whose two first terms on the r.l.s. vanish owing to (1.8). For checking (2.10) we first notice that (2.9) stems from making $\varphi_1 = \epsilon^\alpha$, $\varphi_2 = \epsilon^\beta$ in (1.6), taking account of (2.1) and (2.3). Now we have, from (2.9)

$$(2.11) \qquad \sum_{\alpha,\beta=1}^{r} c_{\beta\alpha}^k (-1)^{\partial e_\alpha}(\epsilon^\alpha \wedge \epsilon^\beta)(e_i, e_j)$$

$$= \sum_{\alpha,\beta=1}^{r} c_{\beta\alpha}^k \{(-1)^{\partial e_i \partial e_j} \delta_i^\alpha \delta_j^\beta - \delta_j^\alpha \delta_i^\beta\} 1_A$$

$$= (-1)^{\partial e_i \partial e_j} c_{ji}^k - c_{ij}^k = -2c_{ij}^k 1_A \qquad \blacksquare$$

Proof of the Proposition. We prove the first relation (2.5) by showing that its r.h.s., say δ_0^1, is a grade-one derivation of $\Lambda^*(L, A)$ wich coincides with δ_0 in grade 0 and 1 (hence everywhere). We have indeed, for $\alpha \in \Lambda(L, A)^\bullet, \beta \in \Lambda(L, A)$, using (1.11):

$$\delta_0^1(\alpha \wedge \beta) = \sum_{i,j=1}^{r} \frac{(-1)^{\partial e_i}}{2} c_{ji}^k \epsilon^i \wedge \epsilon^j \wedge \{(i(e_k)\alpha \wedge \beta + (-1)^{(1+\partial e_k)\partial\alpha} \alpha \wedge i(e_k)\beta\}$$

$$= \left\{\sum_{i=1}^{r} \frac{(-1)^{\partial e_i}}{2} c_{ji}^k \epsilon^i \wedge \epsilon^j \wedge i(e_k)\alpha\right\} \wedge \beta$$

$$(2.12) \qquad + (-1)^{(1+\partial e_k)\partial\alpha + \partial\alpha\,\partial e_k} \alpha \wedge \left\{\sum_{i,j=1}^{r} (-1)^{\partial e_i} c_{ji}^k \epsilon^i \wedge \epsilon^j \wedge i(e_k)\beta\right\}$$

$$= (\delta_0^1 \alpha) \wedge \beta + (-1)^{\partial\alpha} \alpha \wedge (\delta_0^1 \beta)$$

where we used the fact that

$$(2.13) \qquad \partial(c_{ji}^k \epsilon^i \wedge \epsilon^j) = \partial_0 \epsilon^k + \partial e_i + \partial e_j + 1 + \partial_0 \epsilon^i + 1 + \partial_0 \epsilon^j,$$

$$= \partial e_k$$

and we have, on the other hand, that δ_0^1 vanishes on $\Lambda^0(L, A)$ since $i(\xi)$ does, $\xi \in L$; and that it acts as follows on $\Lambda^1(L, A)$ owing to (1.10), (2.1):

$$\delta_0^1 \epsilon^\ell = \sum_{i,j,k=1}^{r} \frac{(-1)^{\partial e_i}}{2} c_{ij}^k \epsilon^i \wedge \epsilon^j \wedge [i(e_k)\epsilon^\ell = (-1)^{\partial e_k(1+\partial e_\ell)} \delta_k^\ell] \tag{2.14}$$

$$= \sum_{i,j=1}^{r} \frac{(-1)^{\partial e_i}}{2} c_{ji}^k \epsilon^i \wedge \epsilon^j$$

identical with (2.7).

We now prove the second relation by the same technique, by showing that its r.h.s., say $d^1\wedge$, is a grade one derivation of $\Lambda^*(L, A)$ coinciding with $d\wedge$ in grade zero and one: indeed we have, on the one hand, for $\alpha \in \Lambda^*(L, A)^\bullet$, $\beta \in \Lambda^*(L, A)$, and each of the summands of $d^1\wedge$, using the derivation property of $d(e_i)$, and (2.3):

$$\begin{aligned}(\epsilon^i\wedge)\, d(e_i)(\alpha\wedge\beta) &= \epsilon^i \wedge \{d(e_i)\alpha \wedge \beta + (-1)^{\partial e_i \partial\alpha}\alpha \wedge d(e_i)\beta\} \\ &= \epsilon^i \wedge (d(e_i)\alpha)\wedge\beta + (-1)^{(\partial e_i + \partial \epsilon_i)\partial\alpha}\alpha\wedge\epsilon^i\wedge d(e_i)\beta \\ &= \{(\epsilon^i\wedge)\, d(e_i)\alpha \wedge \beta + (-1)^{\partial\alpha}\alpha\wedge(\epsilon^i\wedge)\, d(e_i)\beta\}\end{aligned} \tag{2.15}$$

and, on the other hand, on $a \in \Lambda^0(L, A)^\bullet = A^\bullet$

$$d^1 \wedge a = \sum_{i=1}^{r} (-1)^{\partial e_i}\epsilon^i \wedge (e_i a) = \sum_{i=1}^{r} (-1)^{\partial e_i + \partial\epsilon^i\partial(e_i a)}(e_i a)\wedge\epsilon^i \tag{2.16}$$

hence, for $\xi \in L^\bullet$, using the first line in (1.6), the second line in (2.2), (2.3), (1.3) and the first line (1.9)

$$\begin{aligned}(d^1\wedge a)(\xi) &= \sum_{i=1}^{r} (-1)^{\partial e_i + \partial\epsilon^i\partial(e_i a) + \partial(e_i a)}(e_i a)\epsilon^i(\xi) \\ &= \sum_{i=1}^{r} (-1)^{\partial e_i + \partial e_i \partial(e_i a) + \partial(e_i a)(\partial e_i + \partial\xi)}\epsilon^i(\xi)(e_i a) \\ &= \sum_{i=1}^{n} (-1)^{\partial e_i + \partial(e_i a)\partial\xi + \partial e_i(1+\partial\xi)}\xi^i(e_i a) \\ &= (-1)^{\partial a\partial\xi}(\xi a) = (d\wedge a)(\xi)\end{aligned} \tag{2.17}$$

whilst we have on $\varphi \in \Lambda^1(L, A)^\bullet$, for $(\xi_1, \xi_2) \in L^\bullet$, using the second line in (1.6), (1.21), the second line in (2.2), (1.3) and the second line in (1.9)

$$(2.18)\qquad (d^1 \wedge \varphi)(\xi_1, \xi_2) = \sum_{i=1}^{r} (-1)^{\partial e_i}\{\epsilon^i \wedge d(e_i)\varphi\}(\xi_1, \xi_2)$$

$$= \sum_{i=1}^{r} (-1)^{(\partial e_i + \partial_0\varphi)\,\partial\xi_1}\,\epsilon^i(\xi_1)(-1)^{\partial e_i} e_i\{\varphi(\xi_2)\}$$

$$- (-1)^{\partial\xi_2(\partial\xi_1 + \partial e_i + \partial_0\varphi)}\epsilon^i(\xi_2)(-1)^{\partial e_i} e_i\{\varphi(\xi_1)\}$$

$$= \sum_{i=1}^{r} (-1)^{(\partial e_i + \partial_0\varphi)\partial\xi_1 + \partial e_i + \partial e_i(1 + \partial\xi_1)}\xi_1^i e_i\{\varphi(\xi_2)\}$$

$$- (-1)^{\partial\xi_2(\partial\xi_1 + \partial e_i + \partial_0\varphi) + \partial e_i + \partial e_i(1 + \partial\xi_2)}\xi_2^i e_i\{\xi(\xi_1)\}$$

$$= (-1)^{\partial_0\varphi\partial\xi_1}\xi_1\{\varphi(\xi_2)\} - (-1)^{\partial\xi_2(\partial\xi_1 + \partial_0\varphi)}\xi_2\{\varphi(\xi_1)\}$$

$$= (d \wedge \varphi)(\xi_1, \xi_2)$$

completing the proof of (2.5). We now prove the first line in (2.6). It is enough to check equality of the two sides on all products $X\alpha$, $X \in E$, $\alpha \in \Lambda^*(L, A)$: now, we have, by (1.20), since X vanishes on $\Lambda^0(L, E)$:

$$(2.19)\qquad \sum_{i,j,k=1}^{r} \frac{(-1)^{\partial e_i}}{2} c^k_{i,j}(\epsilon^i\bullet)(\epsilon^j\bullet)[i(e_k)(X\alpha) = (-1)^{(1+\partial e_k)\partial X} Xi(e_k)\alpha]$$

$$= \sum_{i,j,k=1} (-1)^{\partial X(1 + \partial e_k + \partial c^k_{ij} + \partial\epsilon^i + \partial\epsilon^j)} X \frac{(-1)^{\partial e_i}}{2} (\epsilon_i\wedge)(\epsilon^j\wedge)i(e_k)\alpha$$

$$= (-1)^{\partial X} X\delta\alpha = \delta_0 X + (-1)^{\partial X} X d\alpha$$

$$= \delta_0(X\alpha)$$

where we used the fact that, from (2.3), (2.4)

$$(2.20)\qquad \partial c^k_{ij} = \partial\epsilon^k([e_i, e_j]) = \partial_0\epsilon_k + \partial e_i + \partial e_j = \partial e_k + \partial\epsilon^i + \partial\epsilon^j$$

The proof of the second line in (2.6) is analogous: we have, using (1.23), and the derivation property of $\delta\wedge$:

$$(2.21)\qquad \sum_{i=1}^{r} (-1)^{\partial e_i}(\epsilon^i\bullet)\,\rho(e_i)\,(X\alpha)$$

$$= \sum_{i=1}^{r} (-1)^{\partial e_i}(\epsilon^i\bullet)[(\rho(e_i)X)\alpha + (-1)^{\partial e_i\partial X} X d(e_i)\,\alpha]$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} (-1)^{\partial e_i} \epsilon^i (\rho(e_i) X) \alpha \\
&+ \sum_{i=1}^{r} (-1)^{(\partial e_i + \partial \epsilon^i)\partial X} X (-1)^{\partial e_i} (\epsilon^i \wedge)\, d(e_i) \alpha \\
&= (\rho \wedge X)\, \alpha + (-1)^{\partial X} X (d \wedge \alpha) \\
&= \rho \wedge (X\alpha).
\end{aligned}
$$

We here use the property

$$\sum_{i=1}^{r} (-1)^{\partial e_i} \epsilon^i \rho(e_i) X = \rho \wedge X \tag{2.22}$$

which is proven as follows. Considering $\Lambda^*(L, E)$ as a left $\Lambda^*(L, A)$ module (cf. footnote 13) relation (1.15) is converted into

$$(\alpha X)(\xi_1, \ldots, \xi_n) = (-1)^{\partial_0 \alpha \partial X} X \alpha(\xi_1, \ldots, \xi_n). \tag{2.23}$$

Using this, the second line of (2.2), and the locality of the E-connection ρ we have, for $\xi \in L^{\bullet}$

$$
\begin{aligned}
&\left\{ \sum_{i=1}^{r} (-1)^{\partial e_i} \epsilon^i \rho(e_i) X \right\} (\xi) \qquad (2.24) \\
&= \sum_{i=1}^{r} (-1)^{\partial e_i + \partial e_i (\partial e_i + \partial X)} (\rho(e_i) X) (\epsilon^i(\xi)) \\
&= \sum_{i=1}^{r} (-1)^{\partial e_i \partial X + (\partial e_i + \partial X)(\partial e_i + \partial \xi)} \epsilon^i(\xi) \rho(e_i) X \\
&= \sum_{i=1}^{r} (-1)^{\partial X \partial \xi + \partial e_i (\partial e_i + \partial \xi) + \partial e_i (1 + \partial \xi)} \xi^i \rho(e_i) X \\
&= (-1)^{\partial X \partial \xi} \rho(\xi) X = (\rho \wedge X)(\xi)
\end{aligned}
$$

(cf. the first line (1.18)). The locality of ∂ was only used in the derivation of (2.22). ∎

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